Mathematical Modeling
Worksheet 6
Insects

Insects in a limited environment.

The study of an insect population in a limited environment revealed that as the population increased the available food per insect declined. In consequence the number of eggs laid N_e decreased in proportion to the population size according to the law $N_e = A - \alpha N_t$ where N_t is the size of the population and A and α are constants. None of the adult insects survive the winter but a fraction λ of the eggs hatch out the following year. Write down a finite difference equations relating N_{t+1} and N_t. Determine the equilibrium value $\overline{N}$ of N_t. Writing $N_t = \overline{N} + n_t$, obtain and solve a difference equation for n_t and hence obtain an expression for N_t assuming $N_t = N_0$ at $t = 0$. Sketch a graph showing the behavior of N_t, when $\alpha\lambda < 1$ and when $\alpha\lambda > 1$.

Predator-Prey Situation

A predator-prey situation occurs with two populations, when encounters between members of the populations benefit the members of one population (predators) but adversely affect the members of the other population (prey). For a simple model assume that the prey has an adequate food supply and that in the absence of interaction there will be a positive rate of increase proportional to the population size. The incidence interactions will depend on the size of both populations and may be taken to be proportional to their product, and the effect will increase the rate of growth of the predator population and decrease the rate of growth of the prey population.

Without a course in DEs, it will be hard for us to solve the pair of differential equations we come across, so I will help you by giving you that the equations eventually simplify down to $\frac{d^2y}{dt^2} = B\frac{dx}{dt}$ and $\frac{d^2x}{dt^2} = -A\frac{dy}{dt}$ where y is the population of the predator and x is the population of the prey and where A and B are positive constants. This in turn simplifies to

$$\frac{d^2y}{dt^2} + ABy = 0$$

which is a famous equation of simple harmonic motion. Find the solution to this for y in terms of t and use that to find a solution for x. Try graphing y and x against t, then consider what a graph of y against x would look like.

Finite difference equations.

The differential equations found above may be represented as finite difference equations in the form

$$X_{t+1} = a_{11}X_t + a_{12}X_tY_t$$

$$Y_{t+1} = a_{21}Y_t + a_{22}X_tY_t$$

Taking $a_{11} = 1.5$, $a_{12} = -0.025$, $a_{21} = 0.75$, $a_{22} = 0.0025$, show that with initial values $X = 100$, $Y = 20$, equilibrium is maintained.
By using a computer and different initial values, verify that the system is not stable.

Competing Species

The differential equation representing growth of a population with a limited food supply requires modification if a second species also competes for the same food resources. In this case the equation becomes...

$$\frac{dN_1}{dt} = r_1N_1(M_1 - N_1 - c_2N_2)$$

where N_1 is the number of species 1, r_1 is the intrinsic growth rate, and M_1 is the saturation population for species 1, N_2 is the number of species 2 and c_2 is the inhibitory factor of species 2 on species 1. The growth rate $\frac{dN_2}{dt}$ is given by a corresponding equation. Taking axes to represent N_1 and N_2, draw lines representing equilibrium conditions for species 1 and 2. Any point (N_1, N_2) represents a possible population configuration. By considering the directions in which this point would tend to move, show under what initial conditions,
a) species 1 would be eliminated
b) species 2 would be eliminated
c) both species would co-exist in stable equilibrium
d) equilibrium would be unstable
(Hint: consider where $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ equal zero.)