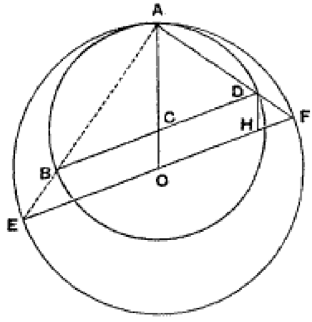
BOOK OF LEMMAS.

Proposition 1.

If two circles touch at A, and if BD, EF be parallel diameters in them, ADF is a straight line.

[The proof in the text only applies to the particular case where the diameters are perpendicular to the radius to the point of contact, but it is easily adapted to the more general case by one small change only.]

Let O, C be the centres of the circles, and let OC be joined and produced to A. Draw DH parallel to AO meeting OF in H.



Then, since $OH = CD = CA$,

and $OF = OA$,

we have, by subtraction,

$$HF = CO = DH.$$

Therefore $\angle HDF = \angle HFD$.

Thus both the triangles CAD, HDF are isosceles, and the third angles ACD, DHF in each are equal. Therefore the equal angles in each are equal to one another, and

$$\angle ADC = \angle DFH.$$

Add to each the angle CDF, and it follows that

$$\angle ADC + \angle CDF = \angle CDF + \angle DFH$$
$$= \text{(two right angles)}.$$

Hence ADF is a straight line.

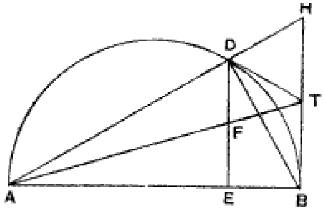
The same proof applies if the circles touch externally*.

Proposition 2.

Let AB be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB, and if AT, DE meet in F,

$$DF = FE.$$

Produce AD to meet BT produced in H. Then the angle ADB in the semicircle is right; therefore the angle BDH is also right. And TB, TD are equal.



Therefore T is the centre of the semicircle on BH as diameter, which passes through D.

Hence $$HT = TB.$$

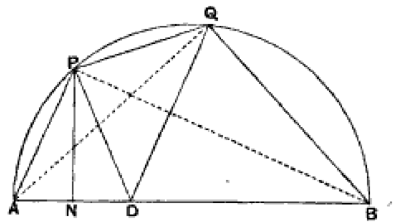
And, since DE, HB are parallel, it follows that $DF = FE$.

* Pappus assumes the result of this proposition in connexion with the ἄρβηλος (p. 214, ed. Hultsch), and he proves it for the case where the circles touch externally (p. 840).

Proposition 3.

Let P be any point on a segment of a circle whose base is AB, and let PN be perpendicular to AB. Take D on AB so that $AN = ND$. If now PQ be an arc equal to the arc PA, and BQ be joined,

BQ, BD shall be equal.*



Join PA, PQ, PD, DQ.

* The segment in the figure of the MS. appears to have been a semicircle, though the proposition is equally true of any segment. But the case where the segment is a semicircle brings the proposition into close connexion with a proposition in Ptolemy's μεγάλη σύνταξις, I. 9 (p. 31, ed. Halma; cf. the reproduction in Cantor's *Gesch. d. Mathematik*, I. (1894), p. 389). Ptolemy's object is to connect by an equation the lengths of the chord of an arc and the chord of half the arc. Substantially his procedure is as follows. Suppose AP, PQ to be equal arcs, AB the diameter through A; and let AP, PQ, AQ, PB, QB be joined. Measure BD along BA equal to BQ. The perpendicular PN is now drawn, and it is proved that $PA = PD$, and $AN = ND$.

Then $\quad AN = \frac{1}{2}(BA - BD) = \frac{1}{2}(BA - BQ) = \frac{1}{2}(BA - \sqrt{BA^2 - AQ^2})$.

And, by similar triangles, $\quad AN : AP = AP : AB$.

Therefore
$$AP^2 = AB \,.\, AN$$
$$= \tfrac{1}{2}(AB - \sqrt{AB^2 - AQ^2}) \,.\, AB.$$

This gives AP in terms of AQ and the known diameter AB. If we divide by AB^2 throughout, it is seen at once that the proposition gives a geometrical proof of the formula

$$\sin^2\frac{\alpha}{2} = \tfrac{1}{2}(1 - \cos\alpha).$$

The case where the segment is a semicircle recalls also the method used by Archimedes at the beginning of the second part of Prop. 3 of the *Measurement of a circle*. It is there proved that, in the figure above,

$$AB + BQ : AQ = BP : PA,$$

or, if we divide the first two terms of the proposition by AB,

$$(1 + \cos\alpha)/\sin\alpha = \cot\frac{\alpha}{2}.$$

Then, since the arcs PA, PQ are equal,

$$PA = PQ.$$

But, since $AN = ND$, and the angles at N are right,

$$PA = PD.$$

Therefore $$PQ = PD,$$

and $$\angle PQD = \angle PDQ.$$

Now, since A, P, Q, B are concyclic,

$$\angle PAD + \angle PQB = (\text{two right angles}),$$

whence $$\angle PDA + \angle PQB = (\text{two right angles})$$
$$= \angle PDA + \angle PDB.$$

Therefore $$\angle PQB = \angle PDB;$$

and, since the parts, the angles PQD, PDQ, are equal,

$$\angle BQD = \angle BDQ,$$

and $$BQ = BD.$$

Proposition 4.

If AB be the diameter of a semicircle and N any point on AB, and if semicircles be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called an ἄρβηλος"; and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P.*

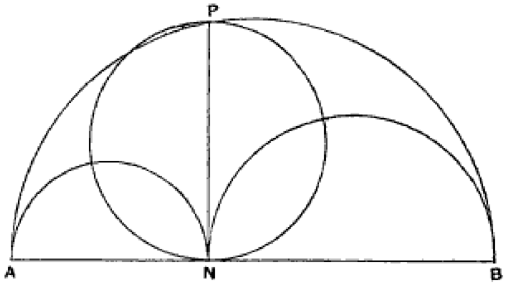
For $$AB^2 = AN^2 + NB^2 + 2AN \cdot NB$$
$$= AN^2 + NB^2 + 2PN^2.$$

But circles (or semicircles) are to one another as the squares of their radii (or diameters).

* ἄρβηλος is literally 'a shoemaker's knife.' Cf. note attached to the remarks on the *Liber Assumptorum* in the Introduction, Chapter II.

Hence

$$(\text{semicircle on } AB) = (\text{sum of semicircles on } AN, NB) + 2\,(\text{semicircle on } PN).$$



That is, the circle on PN as diameter is equal to the difference between the semicircle on AB and the sum of the semicircles on AN, NB, i.e. is equal to the area of the ἄρβηλος.

Proposition 5.

Let AB be the diameter of a semicircle, C any point on AB, and CD perpendicular to it, and let semicircles be described within the first semicircle and having AC, CB as diameters. Then, if two circles be drawn touching CD on different sides and each touching two of the semicircles, the circles so drawn will be equal.

Let one of the circles touch CD at E, the semicircle on AB in F, and the semicircle on AC in G.

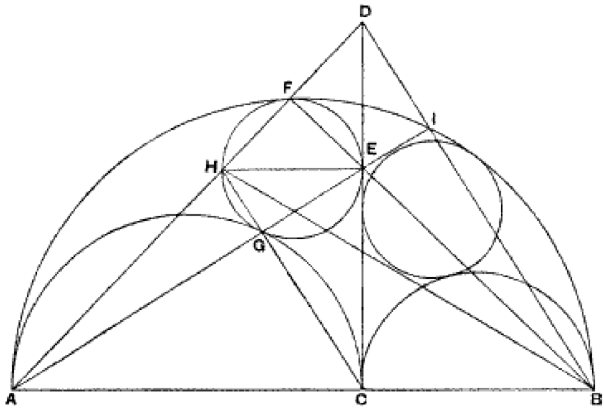
Draw the diameter EH of the circle, which will accordingly be perpendicular to CD and therefore parallel to AB.

Join FH, HA, and FE, EB. Then, by Prop. 1, FHA, FEB are both straight lines, since EH, AB are parallel.

For the same reason AGE, CGH are straight lines.

Let AF produced meet CD in D, and let AE produced meet the outer semicircle in I. Join BI, ID.

Then, since the angles AFB, ACD are right, the straight lines AD, AB are such that the perpendiculars on each from the extremity of the other meet in the point E. Therefore, by the properties of triangles, AE is perpendicular to the line joining B to D.



But AE is perpendicular to BI.

Therefore BID is a straight line.

Now, since the angles at G, I are right, CH is parallel to BD.

Therefore $$AB : BC = AD : DH$$
$$= AC : HE,$$

so that $$AC \cdot CB = AB \cdot HE.$$

In like manner, if d is the diameter of the other circle, we can prove that $$AC \cdot CB = AB \cdot d.$$

Therefore $d = HE$, and the circles are equal*.

* The property upon which this result depends, viz. that

$$AB : BC = AC : HE,$$

appears as an intermediate step in a proposition of Pappus (p. 230, ed. Hultsch) which proves that, in the figure above,

$$AB : BC = CE^2 : HE^2.$$

The truth of the latter proposition is easily seen. For, since the angle CEH is a right angle, and EG is perpendicular to CH,

$$CE^2 : EH^2 = CG : GH$$
$$= AC : HE.$$

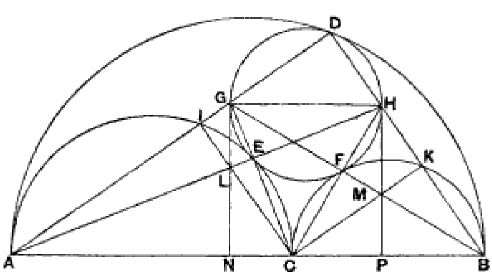
[As pointed out by an Arabian Scholiast Alkauhi, this proposition may be stated more generally. If, instead of one point C on AB, we have two points C, D, and semicircles be described on AC, BD as diameters, and if, instead of the perpendicular to AB through C, we take the radical axis of the two semicircles, then the circles described on different sides of the radical axis and each touching it as well as two of the semicircles are equal. The proof is similar and presents no difficulty.]

Proposition 6.

Let AB, the diameter of a semicircle, be divided at C so that $AC = \frac{3}{2} CB$ [or in any ratio]. Describe semicircles within the first semicircle and on AC, CB as diameters, and suppose a circle drawn touching all three semicircles. If GH be the diameter of this circle, to find the relation between GH and AB.

Let GH be that diameter of the circle which is parallel to AB, and let the circle touch the semicircles on AB, AC, CB in D, E, F respectively.

Join AG, GD and BH, HD. Then, by Prop. 1, AGD, BHD are straight lines.



For a like reason AEH, BFG are straight lines, as also are CEG, CFH.

Let AD meet the semicircle on AC in I, and let BD meet the semicircle on CB in K. Join CI, CK meeting AE, BF

respectively in L, M, and let GL, HM produced meet AB in N, P respectively.

Now, in the triangle AGC, the perpendiculars from A, C on the opposite sides meet in L. Therefore, by the properties of triangles, GLN is perpendicular to AC.

Similarly HMP is perpendicular to CB.

Again, since the angles at I, K, D are right, CK is parallel to AD, and CI to BD.

Therefore $$AC : CB = AL : LH$$
$$= AN : NP,$$
and $$BC : CA = BM : MG$$
$$= BP : PN.$$

Hence $$AN : NP = NP : PB,$$
or AN, NP, PB are in continued proportion*.

Now, in the case where $AC = \frac{3}{2} CB$,
$$AN = \tfrac{3}{2} NP = \tfrac{9}{4} PB,$$
whence $$BP : PN : NA : AB = 4 : 6 : 9 : 19.$$

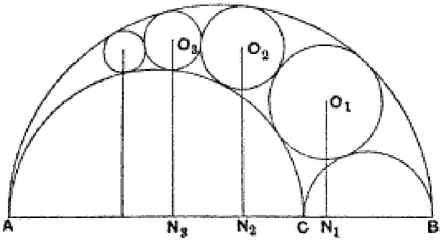
Therefore $$GH = NP = \tfrac{6}{19} AB.$$

And similarly GH can be found when $AC : CB$ is equal to any other given ratio†.

* This same property appears incidentally in Pappus (p. 226) as an intermediate step in the proof of the "ancient proposition" alluded to below.

† In general, if $AC : CB = \lambda : 1$, we have
$$BP : PN : NA : AB = 1 : \lambda : \lambda^2 : (1 + \lambda + \lambda^2),$$
and $$GH : AB = \lambda : (1 + \lambda + \lambda^2).$$

It may be interesting to add the enunciation of the "ancient proposition" stated by Pappus (p. 208) and proved by him after several auxiliary lemmas.



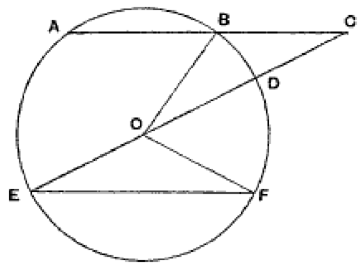
Proposition 7.

If circles be circumscribed about and inscribed in a square, the circumscribed circle is double of the inscribed circle.

For the ratio of the circumscribed to the inscribed circle is equal to that of the square on the diagonal to the square itself, i.e. to the ratio 2 : 1.

Proposition 8.

If AB be any chord of a circle whose centre is O, and if AB be produced to C so that BC is equal to the radius; if further CO meet the circle in D and be produced to meet the circle a second time in E, the arc AE will be equal to three times the arc BD.



Draw the chord EF parallel to AB, and join OB, OF.

Let an ἄρβηλος be formed by three semicircles on AB, AC, CB as diameters, and let a series of circles be described, the first of which touches all three semicircles, while the second touches the first and two of the semicircles forming one end of the ἄρβηλος, the third touches the second and the same two semicircles, and so on. Let the diameters of the successive circles be d_1, d_2, d_3,... their centres O_1, O_2, O_3,... and O_1N_1, O_2N_2, O_3N_3,... the perpendiculars from the centres on AB. Then it is to be proved that

$$O_1N_1 = d_1,$$
$$O_2N_2 = 2d_2,$$
$$O_3N_3 = 3d_3,$$
$$\cdots\cdots\cdots$$
$$O_nN_n = nd_n.$$

Then, since the angles OEF, OFE are equal,

$$\angle COF = 2\angle OEF$$
$$= 2\angle BCO, \text{ by parallels,}$$
$$= 2\angle BOD, \text{ since } BC = BO.$$

Therefore

$$\angle BOF = 3\angle BOD,$$

so that the arc BF is equal to three times the arc BD.

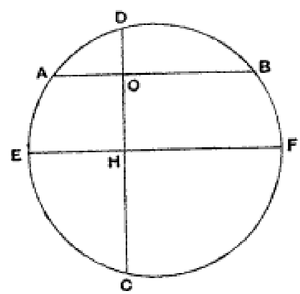
Hence the arc AE, which is equal to the arc BF, is equal to three times the arc BD*.

Proposition 9.

If in a circle two chords AB, CD which do not pass through the centre intersect at right angles, then

$$(arc\ AD) + (arc\ CB) = (arc\ AC) + (arc\ DB).$$

Let the chords intersect at O, and draw the diameter EF parallel to AB intersecting CD in H. EF will thus bisect CD at right angles in H, and



$$(\text{arc } ED) = (\text{arc } EC).$$

Also EDF, ECF are semicircles, while

$$(\text{arc } ED) = (\text{arc } EA) + (\text{arc } AD).$$

Therefore

$$(\text{sum of arcs } CF, EA, AD) = (\text{arc of a semicircle}).$$

And the arcs AE, BF are equal.

Therefore

$$(\text{arc } CB) + (\text{arc } AD) = (\text{arc of a semicircle}).$$

* This proposition gives a method of reducing the trisection of any angle, i.e. of any circular arc, to a problem of the kind known as νεύσεις. Suppose that AE is the arc to be trisected, and that ED is the diameter through E of the circle of which AE is an arc. In order then to find an arc equal to one-third of AE, we have only *to draw through A a line ABC, meeting the circle again in B and ED produced in C, such that BC is equal to the radius of the circle.* For a discussion of this and other νεύσεις see the Introduction, Chapter V.

Hence the remainder of the circumference, the sum of the arcs *AC*, *DB*, is also equal to a semicircle; and the proposition is proved.

Proposition 10.

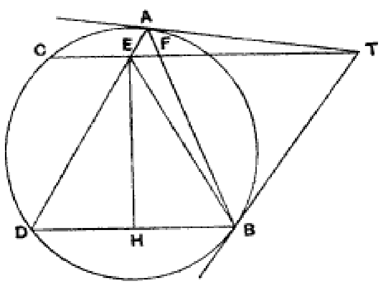
Suppose that TA, TB are two tangents to a circle, while TC cuts it. Let BD be the chord through B parallel to TC, and let AD meet TC in E. Then, if EH be drawn perpendicular to BD, it will bisect it in H.

Let *AB* meet *TC* in *F*, and join *BE*.

Now the angle *TAB* is equal to the angle in the alternate segment, i.e.

$$\angle TAB = \angle ADB$$

$$= \angle AET, \text{ by parallels.}$$



Hence the triangles *EAT*, *AFT* have one angle equal and another (at *T*) common. They are therefore similar, and

$$FT : AT = AT : ET.$$

Therefore

$$ET \,.\, TF = TA^2$$

$$= TB^2.$$

It follows that the triangles *EBT*, *BFT* are similar.

Therefore $\angle TEB = \angle TBF$

$$= \angle TAB.$$

But the angle *TEB* is equal to the angle *EBD*, and the angle *TAB* was proved equal to the angle *EDB*.

Therefore $\angle EDB = \angle EBD$.

And the angles at H are right angles.

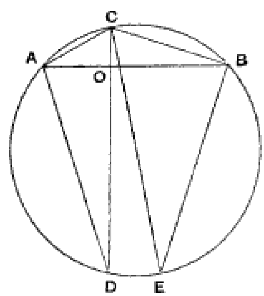
It follows that $BH = HD$*.

Proposition 11.

If two chords AB, CD in a circle intersect at right angles in a point O, not being the centre, then

$$AO^2 + BO^2 + CO^2 + DO^2 = (\text{diameter})^2.$$

Draw the diameter CE, and join AC, CB, AD, BE.



Then the angle CAO is equal to the angle CEB in the same segment, and the angles AOC, EBC are right; therefore the triangles AOC, EBC are similar, and

$$\angle ACO = \angle ECB.$$

It follows that the subtended arcs, and therefore the chords AD, BE, are equal.

* The figure of this proposition curiously recalls the figure of a problem given by Pappus (pp. 836–8) among his lemmas to the first Book of the treatise of Apollonius *On Contacts* (περὶ ἐπαφῶν). The problem is, *Given a circle and two points E, F* (neither of which is necessarily, as in this case, the middle point of the chord of the circle drawn through E, F), *to draw through E, F respectively two chords AD, AB having a common extremity A and such that DB is parallel to EF.* The analysis is as follows. Suppose the problem solved, BD being parallel to FE. Let BT, the tangent at B, meet EF produced in T. (T is not in general the pole of AB, so that TA is not generally the tangent at A.)

Then $\angle TBF = \angle BDA$, in the alternate segment,

$= \angle AET$, by parallels.

Therefore A, E, B, T are concyclic, and

$$EF \cdot FT = AF \cdot FB.$$

But, the circle ADB and the point F being given, the rectangle $AF \cdot FB$ is given. Also EF is given.

Hence FT is known.

Thus, to make the construction, we have only to find the length of FT from the data, produce EF to T so that FT has the ascertained length, draw the tangent TB, and then draw BD parallel to EF. DE, BF will then meet in A on the circle and will be the chords required.

Thus

$$(AO^2 + DO^2) + (BO^2 + CO^2) = AD^2 + BC^2$$
$$= BE^2 + BC^2$$
$$= CE^2.$$

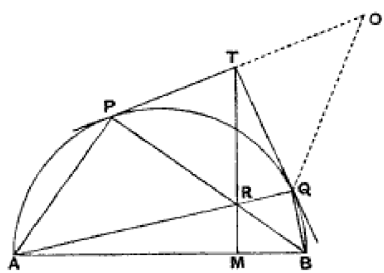
Proposition 12.

If AB be the diameter of a semicircle, and TP, TQ the tangents to it from any point T, and if AQ, BP be joined meeting in R, then TR is perpendicular to AB.

Let TR produced meet AB in M, and join PA, QB.

Since the angle APB is right,

$$\angle PAB + \angle PBA = \text{(a right angle)}$$
$$= \angle AQB.$$



Add to each side the angle RBQ, and

$$\angle PAB + \angle QBA = \text{(exterior)} \angle PRQ.$$

But $\angle TPR = \angle PAB$, and $\angle TQR = \angle QBA$,

in the alternate segments;

therefore $\angle TPR + \angle TQR = \angle PRQ.$

It follows from this that $TP = TQ = TR$.

[For, if PT be produced to O so that $TO = TQ$, we have

$$\angle TOQ = \angle TQO.$$

And, by hypothesis, $\angle PRQ = \angle TPR + TQR.$

By addition, $\angle POQ + \angle PRQ = \angle TPR + OQR.$

It follows that, in the quadrilateral $OPRQ$, the opposite angles are together equal to two right angles. Therefore a circle will go round $OPQR$, and T is its centre, because $TP = TO = TQ$. Therefore $TR = TP$.]

Thus $$\angle TRP = \angle TPR = \angle PAM.$$

Adding to each the angle PRM,

$$\angle PAM + \angle PRM = \angle TRP + \angle PRM$$
$$= \text{(two right angles)}.$$

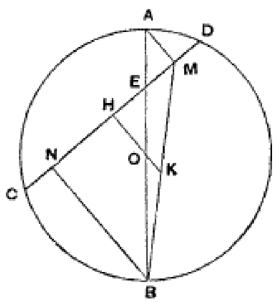
Therefore $$\angle APR + \angle AMR = \text{(two right angles)},$$

whence $$\angle AMR = \text{(a right angle)}^*.$$

Proposition 13.

If a diameter AB of a circle meet any chord CD, not a diameter, in E, and if AM, BN be drawn perpendicular to CD, then

$$CN = DM†.$$



Let O be the centre of the circle, and OH perpendicular to CD. Join BM, and produce HO to meet BM in K.

Then $$CH = HD.$$

And, by parallels,

since $$BO = OA,$$
$$BK = KM.$$

Therefore $$NH = HM.$$

Accordingly $$CN = DM.$$

* TM is of course the polar of the intersection of PQ, AB, as it is the line joining the poles of PQ, AB respectively.

† This proposition is of course true whether M, N lie on CD or on CD produced each way. Pappus proves it for the latter case in his first lemma (p. 788) to the second Book of Apollonius' νεύσεις.

Proposition 14.

Let ACB be a semicircle on AB as diameter, and let AD, BE be equal lengths measured along AB from A, B respectively. On AD, BE as diameters describe semicircles on the side towards C, and on DE as diameter a semicircle on the opposite side. Let the perpendicular to AB through O, the centre of the first semicircle, meet the opposite semicircles in C, F respectively.

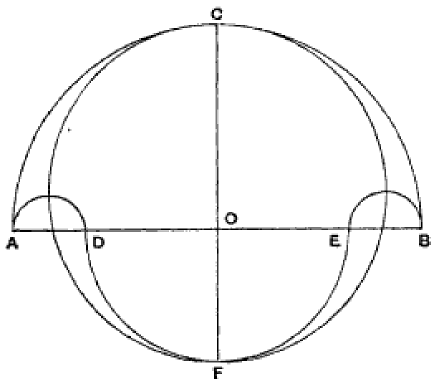
Then shall the area of the figure bounded by the circumferences of all the semicircles ("which Archimedes calls 'Salinon'") be equal to the area of the circle on CF as diameter†.*

By Eucl. II. 10, since ED is bisected at O and produced to A,

$$EA^2 + AD^2 = 2(EO^2 + OA^2),$$

and

$$CF = OA + OE = EA.$$



* For the explanation of this name see note attached to the remarks on the *Liber Assumptorum* in the Introduction, Chapter II. On the grounds there given at length I believe σάλινον to be simply a Graecised form of the Latin word *salinum*, 'salt-cellar.'

† Cantor (*Gesch. d. Mathematik*, I. p. 285) compares this proposition with Hippocrates' attempt to square the circle by means of lunes, but points out that the object of Archimedes may have been the converse of that of Hippocrates. For, whereas Hippocrates wished to find the area of a circle from that of other figures of the same sort, Archimedes' intention was possibly to equate the area of figures bounded by different curves to that of a circle regarded as already known.

Therefore

$$AB^2 + DE^2 = 4(EO^2 + OA^2) = 2(CF^2 + AD^2).$$

But circles (and therefore semicircles) are to one another as the squares on their radii (or diameters).

Therefore

(sum of semicircles on AB, DE)

= (circle on CF) + (sum of semicircles on AD, BE).

Therefore

(area of 'salinon') = (area of circle on CF as diam.).

Proposition 15.

Let AB be the diameter of a circle, AC a side of an inscribed regular pentagon, D the middle point of the arc AC. Join CD and produce it to meet BA produced in E; join AC, DB meeting in F, and draw FM perpendicular to AB. Then

$$EM = (\textit{radius of circle})^*.$$

Let O be the centre of the circle, and join DA, DM, DO, CB.

Now $\angle ABC = \frac{2}{5}$ (right angle),

and $\angle ABD = \angle DBC = \frac{1}{5}$ (right angle),

whence $\angle AOD = \frac{2}{5}$ (right angle).

* Pappus gives (p. 418) a proposition almost identical with this among the lemmas required for the comparison of the five regular polyhedra. His enunciation is substantially as follows. If DH be half the side of a pentagon inscribed in a circle, while DH is perpendicular to the radius OHA, and if HM be made equal to AH, then OA is divided at M in extreme and mean ratio, OM being the greater segment.

In the course of the proof it is first shown that AD, DM, MO are all equal, as in the proposition above.

Then, the triangles ODA, DAM being similar,

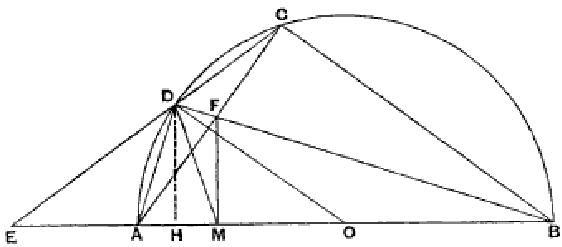
$$OA : AD = AD : AM,$$

or (since $AD = OM$) $\quad OA : OM = OM : MA.$

Further, the triangles FCB, FMB are equal in all respects.

Therefore, in the triangles DCB, DMB, the sides CB, MB being equal and BD common, while the angles CBD, MBD are equal,

$$\angle BCD = \angle BMD = \tfrac{6}{5}\text{(right angle)}.$$



But $\quad \angle BCD + \angle BAD = \text{(two right angles)}$

$$= \angle BAD + \angle DAE$$

$$= \angle BMD + \angle DMA,$$

so that $\quad \angle DAE = \angle BCD,$

and $\quad \angle BAD = \angle AMD.$

Therefore $\quad AD = MD.$

Now, in the triangle DMO,

$$\angle MOD = \tfrac{2}{5}\text{(right angle)},$$

$$\angle DMO = \tfrac{6}{5}\text{(right angle)}.$$

Therefore $\quad \angle ODM = \tfrac{2}{5}\text{(right angle)} = AOD;$

whence $\quad OM = MD.$

Again $\quad \angle EDA = \text{(supplement of } ADC\text{)}$

$$= \angle CBA$$

$$= \tfrac{2}{5}\text{(right angle)}$$

$$= \angle ODM.$$

Therefore, in the triangles EDA, ODM,

$$\angle EDA = \angle ODM,$$

$$\angle EAD = \angle OMD,$$

and the sides AD, MD are equal.

Hence the triangles are equal in all respects, and

$$EA = MO.$$

Therefore $$EM = AO.$$

Moreover $DE = DO$; and it follows that, since DE is equal to the side of an inscribed hexagon, and DC is the side of an inscribed decagon, EC is divided at D in extreme and mean ratio [i.e. $EC : ED = ED : DC$]; "and this is proved in the book of the Elements." [Eucl. XIII. 9, "If the side of the hexagon and the side of the decagon inscribed in the same circle be put together, the whole straight line is divided in extreme and mean ratio, and the greater segment is the side of the hexagon."]