Minimum Vector Rank and Complement Critical Graphs

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Abstract

The minimum rank problem is to find the smallest rank of a collection of matrices which are related to a given graph G. This paper discusses this problem in the context of minimum vector rank for different classes of simple graphs. In particular, the focus is on certain complement critical graphs and the shared structural features of these graphs. The structural similarities allow for determination of the complement criticality and minimum vector rank of a given graph.

1 Introduction

The minimum rank of a graph was the subject of an American Institute of Mathematics workshop in 2006. This topic has roots in various fields and has recently been connected to quantum systems [9]. Hogben states that there seem to be deep connections between minimum rank problems and the control of quantum systems which are only beginning to be explored. In the same paper, Hogben argues the use of minimum rank is a more natural way to parameterize a graph when compared to other notions, such as the chromatic number, since most of these other notions can't fully describe all aspects of a graph.

Our research originates from the work by G. Haynes, C. Park, A. Schaeffer, J. Webster and L. H. Mitchell in 2008 at Central Michigan University. In the paper titled "Orthogonal Vector Coloring"[10] they introduce an orthogonal vector coloring in terms of the chromatic number of a graph.

In contrast with [10], it is more commonly found in [3], [6] and [9] that terminologies used in this area focus on the relation between matrices and the graphs they are describing, which do not allow for explicit description of properties of graph coloring. Therefore, our research began by adopting the coloring approach to investigate graphs in relation to the chromatic number. Further readings in [3], [6], [9] along with [10] inspired our notion of minimum vector rank.

As previously mentioned, this topic is of current interest to many researchers. During the previously mentioned AIM workshop in 2006, "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns" a number of questions were posed throughout the conference. One question in particular has gained much attention: "How large can the sum of the minimum rank of a graph and its complement be?" In other words, what is the upper bound of the following, $\text{mr}(G) + \text{mr}(\overline{G})$? This question has become known as the graph complement conjecture: (GCC) $\mathrm{mr}(G)+\mathrm{mr}(\overline{G}) \leq |G|+2$ in [4] , whose adapted version (GCC_+) we explore later in this paper.

Here arises the notion of complement criticality which is the main focus of our research because it links to the Graph Complement Conjecture. A graph is complement critical if the sum of the minimum vector rank of itself and its complement is greater than the sum of the minimum vector rank of the graph with any vertex removed and the minimum vector rank of its complement with the same vertex removed, which can be phrased $\mathrm{mvr}(G-v) + \mathrm{mvr}(\overline{G-v}) < \mathrm{mvr}(G)+\mathrm{mvr}(\overline{G})$. It is determined in section 4 that the connection between complement criticality and the GCC_+ is that graphs which maximize the GCC_+, $\mathrm{mvr}(G) + \mathrm{mvr}(\overline{G}) \leq |G| + 2$, are complement critical.

In this paper, we first introduce in section 2 basic properties of minimum vector rank with a focus on the graphic aspects of the minimum vector rank. We then provide properties of complement graphs in section 3, specifically the minimum vector rank of several types of complement graphs. This is followed by a section on complement critical graphs found in section 4, in which we explain what it means for a graph to be complement critical. Lastly in section 5, we discuss various classes of complement critical graphs.

2 Definitions and Properties of mvr

A graph is a pair $G = (V, E)$, where V a finite non-empty set of vertices and E is the set of edges, which is often referred to as a simple undirected graph. The order of a graph, G, denoted $|G|$, is the number of vertices of G.

It is decided that the vector representations of a graph in this paper do not include zero vectors, meaning that isolated vertices have no effect on the mr of a graph. Therefore, we will only consider vector representations that do not include zero vectors, which is called non-degenerate. This allows us to give the following definition of mvr.

Definition 2.1 *The minimum vector rank (*mvr*) of a graph G is the minimum rank among non-degenerate vector representations of G.*

As stated above, we define minimum vector rank to be the smallest dimension of a valid vector coloring excluding the zero vectors, which clearly shows the two aspects orthogonal vector coloring: the information encoded in matrices and the characteristics of graphs. The orthogonal representation of a given graph, G, is constructed in the following manner: adjacent vertices are assigned non-orthogonal vectors and the non-adjacent vertices are assigned orthogonal vectors, where zero vectors are not included. To connect our parameter with previous terminologies the relation from [1] is that mvr of a graph is equivalent to the mr_+ of a graph with no isolated vertices. The link between [10]'s use of the vector chromatic number and our minimum vector rank is such that for a given graph G, $\mathrm{mvr}(G) = \chi_v(\overline{G})$. Studying minimum rank problems allows simple results about mr_+ and its connections to Graph Theory. Simple results will be found throughout this paper, in terms of mvr.

The following propositions are simple adaptations from the results found in [10], in terms of mvr.

Proposition 2.2 *Let G be a graph and H an induced subgraph of G. Then, $\mathrm{mvr}(H) \leq \mathrm{mvr}(G)$.*

Proof:

A valid vector coloring of G contains a valid vector coloring of H, thus the minimum vector rank of H will be less than or equal to the minimum vector rank of G. □

Proposition 2.3 *Let K_n be a complete graph on $n \geq 1$ vertices. Then* $\text{mvr}(K_n) = 1$.

Proof:

Every vertex on a complete graph can be colored using the same vector. Therefore, the minimum vector rank of a complete graph is one. □

Proposition 2.4 *If G_1 and G_2 are graphs on the same vertex set, then* $\text{mvr}(G_1 \cap G_2) \leq \text{mvr}(G_1)\text{mvr}(G_2)$.

Proof:

A valid vector coloring of the graphs will have $\text{mvr}(G_1) = n_1$ and $\text{mvr}(G_2) = n_2$. The product of $\text{mvr}(G_1)$ and $\text{mvr}(G_2)$ provides an upper bound, $n_1 n_2$. The vectors $\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_i}$ represent the vertices of G_1, and the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_i}$ represent the vertices of G_2.

The intersection of these two graphs has a valid vector coloring found by taking the tensor product of the corresponding vectors from each graph.

$$(\mathbf{u_i} \otimes \mathbf{v_i}) \cdot (\mathbf{u_j} \otimes \mathbf{v_j}) = (\mathbf{u_i} \cdot \mathbf{u_j})(\mathbf{v_i} \cdot \mathbf{v_j})$$

This shows that the vertices of $(G_1 \cap G_2)$ will be colored by $\mathbf{u_1} \otimes \mathbf{v_1}, \mathbf{u_2} \otimes \mathbf{v_2}, ..., \mathbf{u_i} \otimes \mathbf{v_i}$.

Using the definition of intersection between two graphs, and the rules of a valid vector coloring we can find the bounds created from the tensor product. If two vertices are not adjacent to each other in either G_1 or G_2 their inner product will be zero, giving $(\mathbf{u_1} \otimes \mathbf{v_1}) \cdot (\mathbf{u_2} \otimes \mathbf{v_2}) = 0$. This means that the non-adjacent vertices are assigned vectors which are orthogonal to each other in $G_1 \cap G_2$.

If two vertices are adjacent to each other in both G_1 and G_2 their inner product will not be zero, which gives $(\mathbf{u_1} \otimes \mathbf{v_1}) \cdot (\mathbf{u_2} \otimes \mathbf{v_2}) \neq 0$. This means the adjacent vertices are assigned non-orthogonal vectors in the intersection graph. Thus, the tensor product gives a bound on the dimension of the vectors in the intersection graph as $n_1 n_2$. □

Corollary 2.5 *Let G be a graph with $|G| = n$. Then $n \leq \text{mvr}(G)\text{mvr}(\overline{G})$*

Proof:

It follows from Proposition 3.1, if G_1 and G_2 are complements of each other, then $G_1 \cap G_2$ will be a graph of n independent vertices so that $\text{mvr}(G_1 \cap G_2) = \text{mvr}(nK_1) = n$ where n represents the number of vertices in their intersection. □

3 Definitions and Properties of Graph Complements

The complement of a given graph, G, is defined to be a graph, $\overline{G}$, on the same set of vertices such that there is an edge connecting a set of vertices in $\overline{G}$ if and only if there is not an edge connecting the same set of vertices in G. In the following section, inspired by Leslie Hogben's paper [11], properties of $\mathrm{mvr}(\overline{G})$ are discussed. This will allow investigation of the complement criticality of a graph, which we define in the next section.

Proposition 3.1 *Let G be a graph in which every vertex belongs to at most one cycle. Then* $\mathrm{mvr}(\overline{G}) \leq 3$ *if and only if* $\overline{L_4}$ *(fig. 3, pg. 8) is not an induced subgraph of G.*

The proof for the previous proposition can be found in the Appendix.

Corollary 3.2 *Let T be a tree of order $n \geq 2$, then* $\mathrm{mvr}(\overline{T}) \leq 3$.

Proof:
This is a direct result from Proposition 3.1 because every vertex on a tree belongs to no cycle. Therefore, $\mathrm{mvr}(\overline{T}) \leq 3$. □

Proposition 3.3 *Let T be a tree of order $n \geq 2$, then*

$$\mathrm{mvr}(\overline{T}) = \begin{cases} 2 & \text{if } T = K_{1,n} \\ 3 & \text{otherwise} \end{cases}$$

Proof:
It is known that $\mathrm{mvr}(\overline{K_{1,n}}) = \mathrm{mvr}(K_1 + K_n) = \mathrm{mvr}(K_1) + \mathrm{mvr}(K_n) = 2$. If T is not $K_{1,n}$ (meaning P_4 is an induced subgraph of T), because P_4 is self complementary, the complement of a non-star tree also has P_4 as an induced subgraph whose mvr is 3. Therefore, $\mathrm{mvr}(\overline{T}) \geq 3$ Also, in [1], it is shown that $\mathrm{mvr}(\overline{T}) \leq 3$. Therefore, $\mathrm{mvr}(\overline{T}) = 3$ if T is not $K_{1,n}$. □

Corollary 3.4 *mvr comp. cycle For any cycle C_n,*

$$\mathrm{mvr}(\overline{C_n}) = \begin{cases} 2 & \text{if } n = 4 \\ 3 & \text{otherwise} \end{cases}$$

Proof:
Because a path P_4 is a non-star tree, $\mathrm{mvr}(\overline{P_4}) = 3$ by Proposition 3.3. Also, since a cycle C_{n+1} where $n \geq 4$ can be built on P_4, a cycle on 5 or more vertices fits Proposition .1; and therefore $\mathrm{mvr}(\overline{C_n}) = 3$ for $n \geq 5$.

It is established in [6] that deletion of duplicate vertices does not affect mvr. For $n = 3$, it is observed that $\mathrm{mvr}(\overline{C_3}) = 3$. For $n = 4$, there are 2 pairs of duplicate vertices in $\overline{C_4}$. Therefore, $\mathrm{mvr}(\overline{C_4}) = 2$. □

4 Complement Critical Graphs

We have established some basic results of a graph and a complement graph in terms of mvr. Using these results we can now discuss the notion of complement criticality. This will lead us to explore types of graphs which are complement critical, which then provides a condition for the Graph Complement Conjecture. Complement criticality is the main focus of our research because it links to the Graph Complement Conjecture. A graph G is complement critical if the sum of the minimum vector rank of itself and its complement is greater than the sum of the minimum vector rank of the graph with any vertex removed (represented by $(G - v)$ where v is any vertex) and the minimum vector rank of its complement with the same vertex removed, which can be stated in the following manner:

Definition 4.1 *A graph $G = (V, E)$ is* <u>*critical*</u> *if it satisfies the following inequality for all $v \in V$;*

$$\text{mvr}(G - v) < \text{mvr}(G)$$

Definition 4.2 *A graph $G = (V, E)$ is* <u>*complement critical*</u> *if it satisfies the following inequality for all $v \in V$;*

$$\text{mvr}(G - v) + \text{mvr}(\overline{G - v}) < \text{mvr}(G) + \text{mvr}(\overline{G})$$

The following proposition gives us a way to construct complement critical graphs by combining a critical graph and a complement critical graph, which allows us to build large graphs from simple graphs of which we already have knowledge.

Proposition 4.3 *Let $G = G_1 + \overline{G_2}$ with $\text{mvr}(\overline{G_1}) \leq \text{mvr}(G_2)$. G is complement critical if and only if the following three conditions are satisfied:*

- *G_1 is critical.*
- *G_2 is complement critical.*
- $\text{mvr}(\overline{G_1}) < \text{mvr}(G_2)$.

Note that for a graph $G = G_1 + \overline{G_2}$ where $\text{mvr}(\overline{G_1}) < \text{mvr}(G_2)$,

$$\begin{aligned}
\text{mvr}(G) &= \text{mvr}(G_1) + \text{mvr}(\overline{G_2}) \\
\text{mvr}(\overline{G}) &= \text{mvr}(\overline{G_1} \vee G_2) \\
&= max(\text{mvr}(\overline{G_1}), \text{mvr}(G_2)) \\
&= \text{mvr}(G_2)
\end{aligned}$$

G is complement critical if and only if the mvr goes down for every vertex removed from G_1 or G_2.

Proof:
In the case where v is removed from G_1,

$$\begin{aligned}
\text{mvr}(G - v) &= \text{mvr}(G_1 - v) + \text{mvr}(\overline{G_2}) \\
\text{mvr}(\overline{G - v}) &= max(\text{mvr}(\overline{G_1 - v}), \text{mvr}(G_2)) \\
&= \text{mvr}(G_2) \\
&= \text{mvr}(\overline{G})
\end{aligned}$$

Therefore, $\text{mvr}(G-v)+\text{mvr}(\overline{G-v}) < \text{mvr}(G)+\text{mvr}(\overline{G})$ for all vertices v of G_1 if and only if G_1 is critical.

In the case where v is removed from G_2,

$$\begin{aligned}
\text{mvr}(G-v) &= \text{mvr}(G_1)+\text{mvr}(\overline{G_2-v}) \\
\text{mvr}(\overline{G-v}) &= max(\text{mvr}(\overline{G_1}), \text{mvr}(G_2-v)) \\
&= \text{mvr}(G_2-v)
\end{aligned}$$

Therefore, $\text{mvr}(G-v)+\text{mvr}(\overline{G-v}) < \text{mvr}(G)+\text{mvr}(\overline{G})$ for all vertices v of G_2 if and only if G_2 is complement critical and $\text{mvr}(\overline{G_1}) < \text{mvr}(G_2)$. $\square$

Proposition 4.3 above demonstrates that disconnected complement critical graphs are composed of complement critical components. Therefore, our following discussion is primarily concerned with connected graphs, which leads our interest to others' study in the same area centering a conjecture of graph complement.

The graph complement conjecture was posed as a question at the 2006 AIM workshop. It is about the minimum rank and a graph G and its complement. $\text{mr}(G)+\text{mr}(\overline{G}) \leq |G|+2$ in [4]. Despite the fact that the conjecture is still unproved,there has been some progress over the past several years, and it is now believed that a stronger version (GCC_+) of the conjecture may be true but the question still remains unanswered. We investigate the following version of the conjecture: $\text{mvr}(G)+\text{mvr}(\overline{G}) \leq |G|+2$ in [4].

In the same paper, they define a function $jgap_+ = jmr(G)+jmr(\overline{G})-|G|$, because *jmr* is referred to join minimum rank which is the same as mvr, the $jgap_+$ function can be translated to $jgap_+(G) = \text{mvr}(G)+\text{mvr}(\overline{G})-|G|$. Thus, if $jgap_+(G) \leq 2$ for all graphs, G, the GCC_+ is true.

It can be seen that any graph that maximized the function $jgap_+(G)$ is complement critical. Here we show that if a graph is not complement critical, then the function will not be maximized. If G is not complement critical then when a vertex v is removed from the graph, $\text{mvr}(G-v)+\text{mvr}(\overline{G-v}) = \text{mvr}(G)+\text{mvr}(\overline{G})$ and $|G-v| = |G|-1$; therefore, $jgap_+(G-v) = jgap_+(G)+1$. Thus, $jgap_+(G)$ is not maximized if G is not complement critical.

This leads to the following proposition,

Proposition 4.4 *If a graph G maximizes $jgap_+(G)$, then G is complement critical. In other words, if GCC_+ holds for all complement critical graphs, then it holds for all simple graphs.*

Proposition 4.4 delimitates the non-complement critical graphs satisfying the Graph Complement Conjecture from general graphs.

5 Classes of Complement Critical Graphs

It has been made clear what it means for a graph to be complement critical, the graph must satisfy the complement critical inequality. This allows for classification of different graphs. The following sections discuss families of graphs which satisfy the inequality, and will provide the necessary and sufficient conditions for this satisfaction. The first family is trees, second unicyclic graphs, then necklaces, and lastly books.

5.1 Trees

A tree, denoted T, is defined to be an undirected graph in which any two vertices in T are connected by exactly one simple path. The following discussion is concerned with the complement criticality of trees.

Proposition 5.1 *Any tree T that is not a star is critical.*

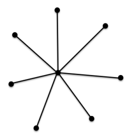

Figure 1: $K_{1,7}$

Proof:

Let T be a non-star tree such that $|T| = n$, then $\text{mvr}(T) = n - 1$ by Lemma 2.5 in [6]. Therefore, $\text{mvr}(T - v) \leq \text{mvr}(T) - 1 = n - 1 - 1 = n - 2$ which demonstrates that T is a critical graph. Because for a tree $T = K_{1,n}$, when the dominating vertex v_d is removed from $K_{1,n}$, $\text{mvr}(K_{1,n}) = \text{mvr}(K_{1,n} - v_d)$ by Theorem 3.2 in [11]. Since the critical inequality asks for the mvr decrease for every single vertex, $K_{1,n}$ is not critical. □

Proposition 5.2 *A graph $G = K_{1,n}$ where $n \geq 1$ is complement critical.*

Proof:

If the vertex being removed from $K_{1,n}$ is the dominating vertex, then $\text{mvr}(K_{1,n} - v) = \text{mvr}(nK_1) = n$ but $\text{mvr}(\overline{K_{1,n} - v}) = \text{mvr}(K_n) = 1 < \text{mvr}(\overline{K_{1,n}})$. Therefore, the complement critical inequality is satisfied if the dominating vertex is removed.

And if the vertex being removed from G is not the dominating vertex (i.e., one of the other equivalent vertices), then $\text{mvr}(K_{1,n} - v) = \text{mvr}(K_{1,n-1}) = n - 1 < \text{mvr}(K_{1,n})$ though $\text{mvr}(\overline{K_{1,n} - v}) = \text{mvr}(K_1 + K_n) = 2 = \text{mvr}(\overline{K_{1,n}})$. Therefore, the complement critical is still satisfied if the non-dominating vertex is removed. □

Corollary 5.3 *to Proposition 4.3 Let T_1, T_2 be trees and have the following two properties: $|T_2| \geq 5$ and T_1 is not a star. Then graph $G = T_1 + \overline{T_2}$ is complement critical.*

Proof:

Let $G_1 = T_1$ and $G_2 = T_2$ in Proposition 4.3. The only requirement for G to be complement critical is that $\text{mvr}(\overline{T_1}) < \text{mvr}(T_2)$. This fits the requirement because T_1 is a non-star and $|T_2| \geq 5$ meaning that $\text{mvr}(\overline{T_1}) = 3$ and $\text{mvr}(T_2) \geq 4 > \text{mvr}(\overline{T_1})$. Thus, G is complement critical. □

5.2 Unicyclic Graphs

Following the discussion of Trees, it is natural to look into unicyclic graphs. A unicyclic graph is defined to be a connected graph with exactly one cycle. The size of the cycle is denoted as c and the number of leaves (vertices of degree 1) on a vertex, v, is denoted as l_v.

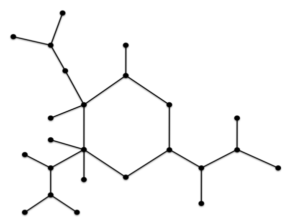

Figure 2: Unicyclic Graph

Proposition 5.4 *Let U be a unicyclic graph, then*

$$\text{mvr}(\overline{U}) = \begin{cases} 4 & \textit{if } \overline{L_4} \textit{ is an induced subgraph} \\ 3 & \textit{otherwise} \end{cases}$$

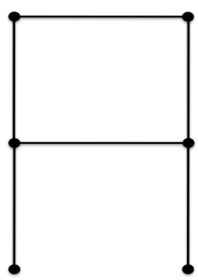

Figure 3: $\overline{L_4}$ (notation adapted from [11])

Proof:

If a unicyclic graph does not have $\overline{L_4}$ as an induced subgraph, then it will either be constructed on cycles of $c = 3$ or $c \geq 5$ or look like one of the three following figures.

In the case where U is on cycles of $c = 3$ or $c \geq 5$, it follows from Corollary **??**, $\text{mvr}(\overline{C}) = 3$ for $c = 3$ or $c \geq 5$. Using the method of Corollary 3.4 from [11], a unicyclic graph U can be constructed from a cycle C by adding one vertex at a time, with the new vertex adjacent to at most one prior vertex. Thus, $\text{mvr}(\overline{U}) = \text{mvr}(\overline{C}) = 3$ given that C represents the cycle on which the unicyclic graph is constructed. Therefore, $\text{mvr}(\overline{U}) = 3$ if $c = 3$ or $c \geq 5$.

Following above, by Corollary 3.4 in [11] $\text{mvr}(\overline{U}) = 4$ will only happen to those unicyclic graphs with $c = 4$. It is easily verified that $\overline{L_4}$ is complement critical since $mvr(\overline{U}) = 4$ and $\text{mvr}(\overline{U - v}) = 3$. Therefore, if a graph contains $\overline{L_4}$ as an induced subgraph, the mvr of that graph is at least 4 because

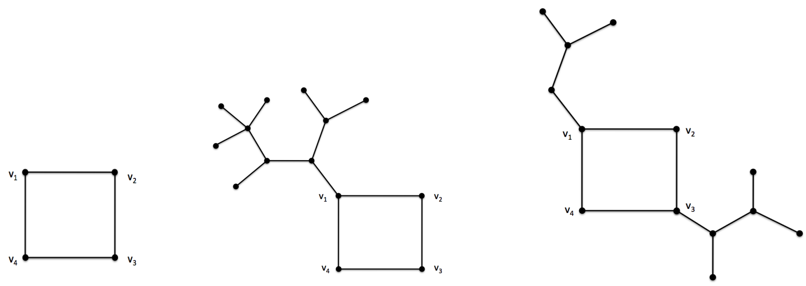

Figure 4: Examples for Non-$\overline{L_4}$ Graphs

$\text{mvr}(\overline{U-v}) \leq \text{mvr}(\overline{U})$. Then, it follows from Corollary 3.4 in [11], $\text{mvr}(\overline{U}) \leq 4$ for $c \geq 3$. Thus, $\text{mvr}(\overline{U}) = 4$ if U contains $\overline{L_4}$ as an induced subgraph.

In the case where a unicyclic graph does not have $\overline{L_4}$ as an induced subgraph, it will look like one of the above three figures. Note that there are duplicate vertices v_2, v_4 in their complements on the cycle; meaning deletion of a duplicate vertices $v_d \in \{v_2, v_4\}$ will not change the mvr. because $(U - v_d)$ is a tree,which gives the following results:

$$\text{mvr}(\overline{U}) \leq 3$$

Thus, any unicyclic graph that does not have $\overline{L_4}$ as an induced subgraph will have $\text{mvr}(\overline{U}) \leq 3$ and any unicyclic graph having $\overline{L_4}$ as an induced subgraph will have $\text{mvr}(\overline{U}) = 4$. □

Lemma 5.5 *The removal of any vertex from a unicyclic graph will have the following results:*

- *If the vertex v is on the cycle, then*

$$\text{mvr}(U - v) = \text{mvr}(U) - \deg(v) + l_v + 2 \tag{1}$$

- *If the vertex v is off the cycle, then*

$$\text{mvr}(U - v) = \text{mvr}(U) - \deg(v) + l_v \tag{2}$$

In particular, if v is a vertex off the cycle then $\text{mvr}(U - v) < \text{mvr}(U)$.

Proof: Let U be a unicyclic graph where $|U| = n$. If the vertex being removed is on the cycle, then the removal will result in a forest with k non-leaf trees and l_v isolated vertices where $|T_i| = n_i$.

$$\begin{aligned}
\text{mvr}(U - v) &= \sum_{i=1}^{k} \text{mvr}(T_i) + l_v \\
&= \sum_{i=1}^{k} (n_i - 1) + l_v \\
&= (n - 1 - l_v) - k + l_v \\
&= n - 1 - k
\end{aligned}$$

Because $k = \deg(v) - l_v - 1$

$$\begin{aligned}\text{mvr}(U - v) &= n - 1 - (\deg(v) - l_v - 1) \\ &= n - \deg(v) + l_v \\ &= (n-2) - \deg(v) + l_v + 2 \\ &= \text{mvr}(U) - \deg(v) + l_v + 2\end{aligned}$$

If the vertex not on the cycle is removed, then $(U - v)$ will be a $(k-1)$-tree forest and a unicyclic graph U' where $|U'| = n'$. Therefore,

$$\begin{aligned}\text{mvr}(U - v) &= \sum_{i=1}^{k-1} \text{mvr}(T_i) + l_v + \text{mvr}(U') \\ &= \sum_{i=1}^{k-1} (n_i - 1) + l_v + (n' - 2) \\ &= \sum_{i=1}^{k-1} (n_i - 1) + (n' - 1) - 1 + l_v \\ &= \sum_{i=1}^{k} (n_i - 1) + l_v - 1 \\ &= (n - l_v - 1) - k + l_v - 1 \\ &= n - 2 - k\end{aligned}$$

Because $k = \deg(v) - l_v$

$$\begin{aligned}\text{mvr}(U - v) &= n - 2 - (\deg(v) - l_v) \\ &= mvr(U) - \deg(v) + l_v\end{aligned}$$

□

All of the results thus far in this section have been leading us to the following result, which gives the necessary and sufficient conditions for a unicylic graph to be complement critical.

Proposition 5.6 *A unicyclic graph U is complement critical if and only if at least one of the following three conditions is true:*

1. *Each vertex on the cycle has at least one non-leaf tree.*
2. *Any unicyclic graph with $\overline{L_4}$ as an induced subgraph.*
3. *A unicyclic graph where $c = 3$ has only one vertex with $l_v \geq 1$ leaves; this graph is denoted as S_n^3 which can be seen in figure 5.*

Proof:

The vertices on a unicyclic graph can be categorized in two cases: those on the cycle and those off the cycle. In the case where the vertices are off the cycle, any removal will always reduce the mvr by Lemma 5.5. If the vertex removed is on the cycle, the mvr may or may not go down depending on the degree of the vertex.

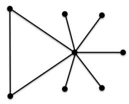

Figure 5: S_8^3

$\Rightarrow$ Assume a unicyclic graph U is complement critical and conditions (1) and (2) don't hold. The only case where a vertex removal will decrease the mvr is when the vertex v is on the cycle. Therefore, this will be the only case we consider.

Assuming condition (1) does not hold, then there is at least one vertex on the cycle with l_v numbers of leaves. By equation (1), if any vertex w on the cycle with no non-leaf tree is removed, $\deg(w) = l_w + 2$ and $\text{mvr}(U - w) = \text{mvr}(U)$.

In addition, since condition (2) does not hold, $\text{mvr}(\overline{U}) \leq 3$ by Proposition 5.4. Because $(U - v)$ is made up of a tree T and l_w number of independent vertices.

$$\begin{aligned}\text{mvr}(\overline{U-w}) &= \text{mvr}(\overline{T + l_w K_1}) \\ &= \text{mvr}(\overline{T} \vee K_{l_w}) \\ &= max(\text{mvr}(\overline{T}), \text{mvr}(K_{l_v})) \\ &= \text{mvr}(\overline{T})\end{aligned}$$

Because U is complement critical and $\text{mvr}(U-w) = \text{mvr}(U)$, then $\text{mvr}(\overline{U-w}) < \text{mvr}(\overline{U}) \leq 3$. Therefore, $\text{mvr}(\overline{U-w}) = \text{mvr}(\overline{T}) < 3$. By Proposition 3.3, then $(U - w)$ must be a star, meaning U must be S_n^3. Thus, if neither condition (1) nor (2) holds, then condition (3) must be true.

$\Leftarrow$Assume condition (3) holds, for any of the 3 vertices w on the cycle of S_n^3 whose degree is equal to $l_w + 2$, then $U = S_n^3$ and we have the following:

$$\begin{aligned}\text{mvr}(U-w) &= \text{mvr}(K_{1,c} + l_w K_1) = 1 + l_w \\ \text{mvr}(\overline{U-w}) &= \text{mvr}(\overline{K_{1,c}} \vee K_{l_w}) = 2\end{aligned}$$

$\text{mvr}(\overline{S_n^3}) = 3$ because there will never be more than 3 independent vertices in $\overline{S_n^3}$, and $\text{mvr}(\overline{S_n^3 - v}) = 2$, the inequality is therefore satisfied.

Assume condition (2) holds, then for any unicyclic graph whose $\text{mvr}(\overline{U}) = 4$ [By proposition], then if the vertex being removed is on the cycle with no tree $\text{mvr}(\overline{U-v}) = 3$ from [lemma 6.7]. Since $\text{mvr}(\overline{U-v} < mvr(\overline{U}) = 4$, the unicyclic graph is complement critical.

Condition (1) is satisfied by equation (1); if there is at least one tree on every vertex on the cycle, then the degree of the vertex being removed is greater than $l_v + 2$, so the mvr will go down and the graph is critical. $\square$

Corollary 5.7 *Cycles are not complement critical.*

Proof:

A cycle fails to satisfy any of the 3 conditions in Proposition 5.6, therefore it is not complement critical. □

5.3 Necklaces

This next type of graph naturally follows from the discussions of trees and unicylic graphs, simply because it is a combination of the two.

Definition 5.8 *A necklace, N, is defined to be a connected graph with in which each vertex appears in at most one cycle.*

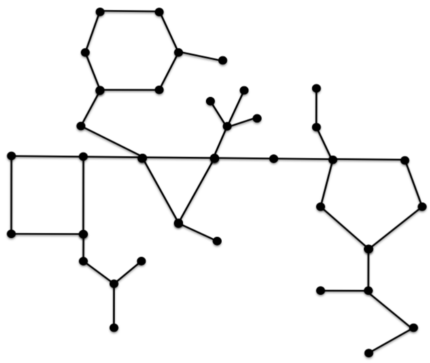

Figure 6: Necklace

Proposition 5.9 *For any necklace N with k cycles,* $\text{mvr}(N) = |N| - k - 1$.

Proof:

Let N be a necklace with k cycles. Because the vertices on the cycle are cut vertices, if one cut vertex from each cycle is removed, N becomes a disconnected graph of independent vertices and t trees. Each of the trees results from the removal of a cut vertex from cycle C_i. Therefore, there are $(k-1)$ cut vertices removed in total. Using Theorem 3.1 from [9] $(k-1)$ times, we get the following result.

$$\begin{aligned}
\text{mvr}(N) &= \sum_{i=1}^{k} \text{mvr}(C_i) + \sum_{i=1}^{t-k} \text{mvr}(T_i) + (t-1) \\
&= \sum_{i=1}^{k} (|C_i| - 2) + \sum_{i=1}^{t-k} (|T_i| - 1) + t - 1 \\
&= (|N| - k - t) + t - 1 \\
&= |N| - k - 1
\end{aligned}$$

□

It is defined in [4] that if a graph $G = \cup_{i=1}^{r} G_i$, where each G_i is connected, then the subgraph H of G is $H = \cup_{|G_i| \geq 1} G_i$ and is called the core of G. The following proposition makes use of the idea of LSEAC graphs Hogben firstly proposed in [11].

Proposition 5.10 *Let v be a vertex on a necklace N, then* $\text{mvr}(N - v) < \text{mvr}(N)$ *unless both following conditions are satisfied:*

- *the vertex v is on a cycle.*
- *the core of $(N - v)$ is connected.*

Proof:

Let N be a necklace with k cycles and v be the vertex removed, then $(N-v)$ will still be several disconnected small necklaces and independent vertices.

When a vertex on the cycle but does not disconnects the core of $(N - v)$ is removed, both both $|N - v|$ and k' is 1 less than the original $|N|$ and k. Therefore, $\text{mvr}(N - v) = (|N| - 1) - (k - 1) - 1 = \text{mvr}(N)$. □

Proposition 5.11 *Let N be a necklace, then*

$$\text{mvr}(\overline{N}) = \begin{cases} 4 & \textit{if } \overline{L_4} \textit{ is an induced subgraph} \\ 3 & \textit{otherwise} \end{cases}$$

Proof:

Because a necklace can be built from a unicyclic graph by repeatly adding a degree 2 vertex joining two pendent vertices that are off the cycle, which fits the conditions in Proposition .1. Therefore, this proof can be achieved by Proposition 5.4.

For any necklace N having $\overline{L_4}$ as an induced subgraph, $\overline{N}$ can be constructed by adding vertices as discussed above from a unicyclic graph containg $\overline{L_4}$, so $\text{mvr}(\overline{N}) = 4$.

If N does not have $\overline{L_4}$ as an induced subgraph, it can be contructed on a unicyclic graph with no $\overline{L_4}$. Thus, $\text{mvr}(\overline{N}) = 3$ in this case. □

Proposition 5.12 *A necklace N is complement critical if and only if one of the following three conditions is true:*

- *every vertex removed disconnects the core of $(N - v)$.*
- *$\overline{L_4}$ is an induced subgraph of N.*
- *$N = S_n^3$.*

Proof:

This is a generalization of Proposition 5.6 with a major change that vertices with non-leaf tree is generalized to vertices disconnects the core of $(N-v)$. Proposition 5.10 clearly shows that this generalization is true for necklace. Thus, everything else follows. □

5.4 Books

The last type of graph we explore is a book, whose definition is inspired by [3]. It can be seen that a book is composed of several cycles with trees. Since we have already examined the properties of those types of graphs earlier, the following section is a compilation of results which are similar to those from earlier in the paper.

Definition 5.13 *For $m \geq 2$, $i \geq 3$, A* <u>*book*</u> *$B(K_1, K_2, \ldots K_m)$ is defined to be a graph in which all cycles in the book $K_1, K_2, \ldots K_m$ share one common edge. The two vertices on the common edge are called the binding vertices. We denote the two binding vertices as v_b and the book as B_m in compressed form.*

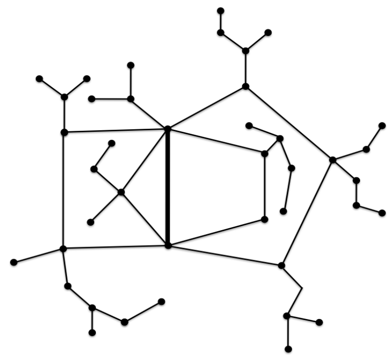

Figure 7: book

Proposition 5.14 *Let B_m be a book, then* $\text{mvr}(B_m) = |B_m| - 2$.

Proof:

For a book of cycles, this is a direct generalization of Proposition 3.11 in [3] because $\text{mvr}(B_m) \geq |B_m| - Z_+(B_m) = |B_m| - 2$. The binding vertices form a positive forcing set for the books. Because a book B_m is constructed on a book of cycles by adding one vertex adjacent to at most one prior vertex at a time, both $|B_m|$ and $\text{mvr}(B_m)$ increase by one with every single vertex added. Thus, the equality still holds for books. □

Proposition 5.15 *Let B_m be a book, then if a vertex is removed from B_m,* $\text{mvr}(B - v) < \text{mvr}(B_m)$ *unless the vertex being removed is a binding vertex, v_b with no non-leaf trees on it.*

Proof:

Let B_m be a book with m pages and B_{m-1} be the book where one page of B_m is removed. There are two types of vertices that can be removed from B_m.

If the vertex being removed is the binding vertex with k non-leaf trees and l_v leaves, then the removal will result in a forest with $(k+1)$ non-leaf trees (not leaves) where $|T_i| = n_i$ and l_v independent vertices. This is similar to our previous result found in unicyclic graphs. Therefore,

$$\begin{aligned}
\text{mvr}(B_m - v_b) &= \sum_{i=1}^{k+1} \text{mvr}(T_i) + l_v \\
&= \sum_{i=1}^{k+1} (|T_i| - 1) + l_v \\
&= (\sum_{i=1}^{k+1} |T_i| + l_v) - (k+1) \\
&= (|B_m| - 1) - (k+1) \\
&= (|B_m| - 2) - k \\
&= \text{mvr}(B_m) - k
\end{aligned}$$

Therefore, it can be easily seen that $\text{mvr}(B_m)$ would decrease as long as $k > 0$(i.e., there are trees on the binding vertex).

The removal of a non-binding vertex gives an induced subgraph of B_m which is still a book together with k trees and l_v independent vertices. Because the new book B'_m is separated from the trees and independent vertices, $\text{mvr}(B_m - v)$ can be calculated by adding up the mvrof each component.

$$\begin{aligned}
\text{mvr}(B_m - v) &= \text{mvr}(B'_m) + \sum_{i=1}^{k} \text{mvr}(T_i) + l_v \\
&= (|B'_m| - 2) + \sum_{i=1}^{k} (|T_i| - 1) + l_v \\
&= (|B'_m| + \sum_{i-1}^{k} |T_i| + l_v) - (k+2) \\
&= (|B_m| - 1) - (k+2) \\
&= \text{mvr}(B_m) - k - 1 \\
&< \text{mvr}(B_m)
\end{aligned}$$

□

Proposition 5.16 *For a book of unicyclic graphs B_m with $m \geq 2$,*

$$\text{mvr}(\overline{B_m}) = \begin{cases} 4 & \textit{if } C_4 \textit{ is an induced subgraph} \\ 3 & \textit{otherwise} \end{cases}$$

Proof:

Let B be a book with $m \geq 2$. If it has C_4 as an induced subgraph, then B_m will have one of the following three graphs: domino, $\overline{L_4}$, $\overline{P_5}$ as an induced subgraph as well. For any of the three graphs, the vectors assigned to any three vertices are linearly independent which satisfies condition (3) and (4) in Theorem 2.2 from [11]. Therefore, $\text{mvr}(\overline{B_m}) \leq 4$. Also, B_m has at least one of the following induced subgraphs:domino, $\overline{L_4}$, $\overline{P_5}$ and hence $\overline{B_m}$ will have at least one of the following as an induced subgraph: co-domino, L_4, P_5. Therefore, mvr($\overline{B_m}$) is at least 4 because any of the above graphs has mvrof 4. Thus, mvr($\overline{B_m}$) = 4 if B_m has C_4 is an induced subgraph.

In order to show that when B_m does not have C_4 as an induced subgraph mvr($\overline{B_m}$) = 3, we are using the result from appendix that Given a book B_m of the graphs $C_1, C_2, \ldots C_k$ with $i = 3$ or $n \geq 5$. Then mvr($\overline{B_m}$) = $\max_i$ mvr($\overline{C_i}$) = 3.. □

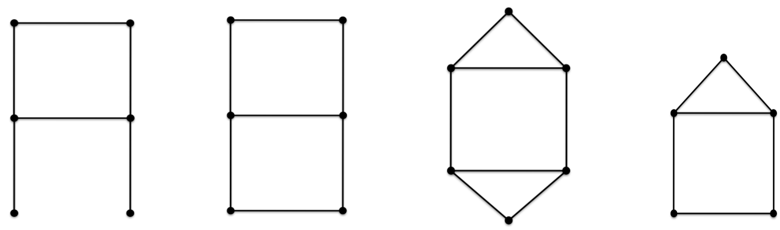

Figure 8: $\overline{L_4}$, Domino, Co-domino, $\overline{P_5}$

By Proposition 5.15, the removal of a non-binding vertex will always decrease the mvr; due to this, the removal of either binding vertex is of interest in terms of graph complement criticality.

Proposition 5.17 *For a book B_m, the removal of one binding vertex will give the following results,*

$$\text{mvr}(\overline{B_m - v_b}) = \begin{cases} 2 & \text{if } B_m = B_m^3 \\ 3 & \text{otherwise} \end{cases}$$

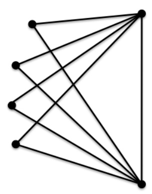

Figure 9: B_m^3

Proof:

Let $B_m = B_m^3$ be a book with $m \geq 2$. The removal of v_b gives a star, therefore, mvr($\overline{B_m^3 - v_b}$) = 2.

In the case where $B_m \neq B_m^3$, $(B_m - v_b)$ will be a forest and some independent vertices. Because there must exist trees that $T \neq K_{1,n}$ in the forest; mvr($\overline{B_m - v_b}$) = max_imvr($\overline{T_i}$) = 3 by Proposition 3.3. □

Proposition 5.18 *A generalized book B_m is complement critical if and only if it satisfies one of the following conditions:*

- *There are non-leaf trees on both binding vertices.*

- C_4 *is an induced subgraph of* B_m.
- $B_m = B_m^3$

Proof: According to Proposition 5.15, the only interesting case is when a binding vertex v_b is removed. Thus, the following discussion will only consider this case. In Section 2.3 unicyclic graphs, we have a result which can be extended to books B_m.

$\Rightarrow$ Here, we are showing that if none of the conditions holds, then B_m is not complement critical. It follows from Proposition 5.16 and 5.17, because C_4 is not an induced subgraph of B_m and $B_m \neq B_m^3$, then $\mathrm{mvr}(\overline{B_m}) = \mathrm{mvr}(\overline{B_m - v}) = 3$. Also, by Proposition5.15, $\mathrm{mvr}(B_m) = \mathrm{mvr}(B_m - v)$Thus, the book B_m is not complement critical.

$\Leftarrow$ In the case where condition (1) holds, it is obvious that the complement critical inequality is satisfied because $\mathrm{mvr}(\overline{B_m}) > \mathrm{mvr}(\overline{B_m - v})$ by Proposition 5.15. In the case where condition (2) holds,by Proposition 5.16 and 5.17, $\mathrm{mvr}(\overline{B_m}) = 4 > \mathrm{mvr}(\overline{B_m - v_b}) = 3$. In the case where condition (3) holds, by Proposition 5.16 and 5.17, $\mathrm{mvr}(\overline{B_m^3}) = 3 > \mathrm{mvr}(\overline{B_m^3 - v_b}) = 2$. Therefore, if a book saitisfies any of the three conditions, it is complement critical. □

6 Conclusion

In this paper, we introduced basic properties of minimum vector rank and focused specifically on the minimum vector rank of complement critical graphs. We found a link between complement critical graphs and the GCC_+, which allowed for testing the conjecture by studying complement critical graphs only. We provided a list of general properties of complement graphs and provided explicit examples of families of complement critical graphs which are structurally similar. Then, we explained the necessary and sufficient conditions for these graphs to be complement critical.

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Appendix on Graph Complements by M. Nathanson

While bounds are known for the mvr of the complement of specific graphs (as established in [11]), we will need stronger results on a wider class of graphs for our results on complement critical graphs. Proposition 3.1 was stated in Section 4. Its proof is given here as a consequence of the following construction:

Proposition .1 *Let H be a graph with vertices v and w such that*

- *v and w are pendant vertices* $(\deg(v) = \deg(w) = 1)$
- *v and w are not adjacent to each other; and they are not adjacent to the same vertex.*

Let G be the graph formed from H by adding a new vertex u of degree 2 which is adjacent to v and w.

If there is an orthogonal representation of $\overline{H}$ in $R^d, d \geq 3$ which assigns distinct unit vectors to each vertex, then there is such an orthogonal representation of $\overline{G}$ in R^d.

Proof:

Assume that there exists a vector labeling of the vertices of $\overline{H}$ in R^d that assigns distinct unit vectors to each vertex. Abusing notation, let v and w

represent the vectors corresponding to the vertices v and w; let y_1 and y_2 represent the vectors corresponding to the respective neighbors of v and w; and let $y_3, \ldots y_k$ represent the vectors corresponding to the remaining vertices of H.

For any real number t, define

$$u_t = tv + w$$
$$v_t = u_t \times v'$$
$$w_t = u_t \times w'$$

If $d > 3$, then the cross-product is taken inside any fixed three-dimensional subspace which contains v and w. (By assumption, such a subspace not orthogonal to either y_1 or y_2.)

We wish to show that there exists a value of t for which $\{u_t, v_t, w_t, y_1, \ldots, y_k\}$ is an orthogonal representation of $\overline{G}$.

By construction, for all t,

$$\langle u_t, v_t\rangle = \langle u_t, w_t\rangle = \langle v_t, y_1\rangle = \langle w_t, y_2\rangle = 0$$

Since these are the only non-neighbors of our vertices u, v, w, we require only that no other inner products are zero. Specifically, we need to show that there exists a value of t such that:

1. $\langle u_t, y_i\rangle \neq 0$ for all $i = 1 \ldots k$
2. $\langle v_t, y_i\rangle \neq 0$ for $i \neq 1$ and $\langle w_t, y_i\rangle \neq 0$ for $i \neq 2$
3. $\langle v_t, w_t\rangle \neq 0$

The result is proved as long as none of these functions of t is identically zero.

Looking at line (1), $\langle u_t, y_i\rangle$ is identically zero if and only if $\langle v, y_i\rangle = \langle w, y_i\rangle = 0$, and neither of these conditions holds. In line (2), $\langle v_t, y_i\rangle = \det(u_t, v', y_i) = 0$ if and only if $y_i \in Span(v, y_1) \cap Span(w, y_1) = Span(y_1)$, which occurs only if $i = 1$. The second part of (2) is similarly shown.

Finally, the left side of equation (3) is a quadratic function, which we can write as $f(t) = \langle v_t, w_t\rangle$. Using the fact that $\langle v, y_1\rangle = 0$, we calculate $f''(t) = 2\langle v, v\rangle\langle y_1, y_2\rangle$. If $f''(t) \neq 0$, then f is a non-degenerate quadratic with only two zeroes. But if $f''(t) = 0$, then $\langle y_1, y_2\rangle = 0$ and $f(t)$ is actually a linear function with slope $\langle v, y_2\rangle\langle w, y_1\rangle \neq 0$. In either case, $f(t) \neq 0$ for almost all t.

Since each condition excludes finitely many values of t, there exists a value of t which makes $\{u_t, v_t, w_t, v', w', y_1, \ldots, y_k\}$ an orthogonal representation of $\overline{G}$. □

The previous result will allow us to build up orthogonal representations vertex by vertex. The next lemma is in a similar spirit:

Lemma .2 *Let H be a graph with pendant vertex w. Let G be the graph formed from H by adding 2 new vertices u and v which are adjacent to each other and also to w but have no other neighbors. (So, the induced graph $G[u, v, w] = K_3$ and $G[u, v, y]$ is disconnected for all vertices $y \notin \{u, v, w\}$.)*

If there is an orthogonal representation of $\overline{H}$ in R^3, then there is an orthogonal representation of $\overline{G}$ in R^3.

Observation 1.3 in [11] guarantees that there exists a unit vector u_0 in R^3 which is orthogonal to w but not to any other vector in the representation. Let $v_0 = u_0 \times w$. As in the previous proof, we define

$$\begin{aligned} u_t &= u_0 + tv_0 \\ v_t &= tu_0 - v_0 \end{aligned}$$

It is clear that for all t, the vectors u_t, v_t and w are mutually orthogonal. And also that for any other vector y in the orthogonal representation, $\langle y, u_t \rangle$ and $\langle y, v_t \rangle$ are not identically zero because $\langle y, u_0 \rangle \neq 0$.

Therefore, we can find a value of t which extends the orthogonal representation of H to G.

We use these two results in the proof of the main proposition.

.1 Proof to Proposition 3.1

Note that if G is a disconnected graph with components G_i, then $\text{mvr}(\overline{G}) = \max_i \text{mvr}(\overline{G_i})$ and $\overline{L_4}$ is an induced subgraph of G if and only if it is a subgraph of some G_i. So, if the proposition is true for connected graphs, then it must be true for all simple graphs.

We also note that one direction of the proof is clear: If $\overline{L_4}$ is an induced subgraph of G, then $\text{mvr}(\overline{G}) \geq \text{mvr}(L_4) = 4$. The proof of the converse comes from the fact that any such graph G can be built up one vertex at a time using a small number of operations.

Proof by strong induction: The proposition evidently holds for K_2. Assume that the proposition holds for all connected graphs with fewer than $n > 2$ vertices in which every vertex belongs to at most one cycle. Now, consider such a graph G with n vertices such that $\overline{L_4}$ is not an induced subgraph. Note that every induced subgraph of G also satisfies the hypotheses of the proposition.

If $\overline{G}$ contains a pair of duplicate vertices u and v, then let $H = G - v$. Since duplicate vertices don't change the mvr, $\text{mvr}(\overline{G}) = \text{mvr}(\overline{H}) \leq 3$ by the inductive hypothesis.

Now, assume that $\overline{G}$ contains no duplicate vertices . Since G does not have $\overline{L_4}$ as an induced subgraph, this implies that G contains no 4-cycle.

If G contains a pendant vertex v, then Theorem 2.1 in [11] asserts that $\text{mvr}(\overline{G}) = \text{mvr}(\overline{G - v})$ as long as any two vectors in the representation of $\overline{G - v}$ are linearly independent, which is equivalent to having no duplicate vertices. Thus, $\text{mvr}(\overline{G}) \leq 3$ by the inductive hypothesis.

Assume now that G contains no pendant vertices. Thus, G is simply a bunch of cycles connected by paths. Note that we can associate a tree T with G by collapsing each cycle of G into one vertex of T. Because a tree has at least two leaves, we see that G has at least two cycles which are attached to the rest of the graph by a single edge. Let us consider one such cycle C attached to G at w. We know that it is not a 4-cycle. If it is a 3-cycle on (u, v, w), let $H = G - u - v$ and apply Lemma .2. If C is a cycle of length greater than four, let u be a vertex on C at distance 2 from w. Let $H = G - u$ and apply Proposition .1. In either case, we see that $\text{mvr}(\overline{G}) = \text{mvr}(\overline{H}) \leq 3$ by the inductive hypothesis.

Thus, $\text{mvr}(\overline{G}) \leq 3$ if and only if $\overline{L_4}$ is not an induced subgraph of G.

□